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Normal modes of a continuous system with quadratic and cubic non-linearities

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Abstract

A power series solution is presented for the free vibrations of simply supported beams resting on elastic foundation having quadratic and cubic non-linearities. The time-dependence is assumed harmonic and the problem is posed as a non-linear eigenvalue problem. The spatial variable is transformed into an independent variable that satisfies the boundary conditions. This permits a power series expansion of the beam motion in terms of the new variable. A recurrence relation is obtained from the governing equation and used in conjunction with the Rayleigh energy principle to compute the natural frequencies. The results show that, for a first order approximation, only the lower frequencies and first mode shape are significantly affected by the cubic non-linearity.

1. Introduction

Knowledge of the normal modes and frequencies of non-linear continuous systems constitutes an important part in the dynamic analysis under loading conditions. At present, as for most non-linear systems, no exact method of solution is available for such problems and resort is normally made to some approximate method. In one approach, the motion is assumed to be a combination of the linear normal modes whose contributions are represented by time-dependent generalized co-ordinates. The Galerkin procedure is then employed to obtain a set of non-linear coupled ordinary differential equations for the generalized co-ordinates. The discretized equations are normally solved approximately by a perturbation technique [1,2] or the center manifold reduction method [3]. Recently, the power series method has been used to obtain a solution of the generalized co-ordinates for a beam resting on a non-linear foundation [4]. The equation of motion was first discretized by using the Galerkin procedure. The time-dependent generalized

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co-ordinates were obtained by transforming the time variable into an oscillating time which transformed the discretized equations into a form solvable by the power series method.

In another approach, the time-dependence is assumed harmonic and either the harmonic balance principle or the Ritz method is used to produce a non-linear eigenvalue problem. Szemplinska-Stupnicka [5] employed the harmonic balance method in conjunction with the generalized Ritz method to determine the mode shapes of some non-linear beam systems and considered the effects of non-linear boundary conditions.

More recently, Benamar et al. [6,7] assumed harmonic motion to compute the fundamental mode and frequency of beams and rectangular plates at large amplitudes. The plate motion was assumed to consist of products of the first five beam mode shapes and resulted in a set of 24 non-linear algebraic equations which were solved numerically.

This work adopts the second approach to study the free vibrations of a simply supported beam resting on non-linear foundation and uses the Ritz method to reduce the governing equation to a non-linear eigenvalue problem. The spatial variable is first transformed into a new variable that satisfies the beam boundary conditions. The transformed eigenvalue problem is then solved with the aid of the Rayleigh energy principle.

2. Formulation

The problem treated here is the free vibrations of a linear Euler–Bernoulli simply supported beam resting on a foundation with quadratic and cubic non-linearities.The governing equation of motion is given by

$$\frac{\partial^4 w}{\partial x^4} + \alpha_1 w + \beta_1 w^2 + \gamma_1 w^3 + \frac{\partial^2 w}{\partial t^2} = 0, \tag{1}$$

where $w(x,t)$ is the beam transverse displacement at position x and time t and the constants $\alpha_1, \beta_1, \gamma_1$ characterize the stiffness of the foundation. For moderate vibration amplitudes, a first order approximate solution of the form

$$w(x,t) = u(x)\cos(\omega t) \tag{2}$$

can be used, where $u(x)$ is the normal mode of vibration and ω is the vibration frequency. When this approximation is substituted in Eq. (1), a residual error $\varepsilon(x,t)$ results which is minimized following the Ritz method by requiring that

$$\int_0^{2\pi} \varepsilon(x,t)\cos(\omega t)\,\mathrm{d}(\omega t) = 0. \tag{3}$$

This process yields the non-linear eigenvalue problem

$$\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} + \alpha u + \beta u^2 + \gamma u^3 - \omega^2 u = 0 \tag{4}$$

subject to the boundary conditions $u = \mathrm{d}^2u/\mathrm{d}x^2 = 0$ at $x = 0, 1$ and the constants $\alpha = \alpha_1$, $\beta = 0$, $\gamma = \frac{3}{4}\gamma_1$. It is noted here that, for a first order approximation, the quadratic non-linearity has no effect on the normal modes of vibration.

To facilitate the use of the power series method for this problem, it is convenient to transform the independent variable x into a new variable η that satisfies the boundary conditions as follows:

$$\eta = \sin n\pi x, \tag{5}$$

where n is the mode number. Upon using Eq (5) in Eq. (4), the transformed eigenvalue problem becomes

$$(1-\eta^2)^2\frac{d^4u}{d\eta^4} - 6\eta(1-\eta^2)\frac{d^3u}{d\eta^3} + (7\eta^2-4)\frac{d^2u}{d\eta^2} + \eta\frac{du}{d\eta} + (\alpha u + \beta u^2 + \gamma u^3 - \omega^2 u)/(\eta\pi)^4 = 0. \tag{6}$$

According to the theory of ordinary differential equations [8], Eq. (6) has one ordinary point at $\eta = 0$ and two regular singular points at $\eta = \pm 1$. It is appropriate to expand $u(\eta)$ about the ordinary point as follows:

$$u(\eta) = a_1 + a_2\eta + a_3\eta^2 + a_4\eta^3 + \cdots = \sum_{k=1}^{\infty} a_k\eta^{k-1}, \tag{7}$$

where the coefficients a_i are constants to be determined. It is noted that all the boundary conditions on u, as given by Eq. (7), are satisfied if one sets $a_1 = a_3 = 0$. Substituting Eq. (7) in Eq. (6) and introducing a shift of indices, where necessary, so that all terms have the same power form, one obtains

$$\sum_{k=1}^{\infty}[C_1a_{k+4} - C_2a_{k+2} + C_3a_k + (\alpha a_k + \beta b_k + \gamma c_k - \omega^2 a_k)l(\eta\pi)^4]\eta^{k-1} = 0, \tag{8}$$

where

$$C_1 = k(k+1)(k+2)(k+3),$$
$$C_2 = 2k(k+1)(k^2+1),$$
$$C_3 = (k-1)((k-2)(k^2-k+1)+1).$$

The coefficients b_k, c_k are those of the non-linear terms u^2, u^3, respectively, which result from repeated multiplication of Eq. (7) and can be computed once the coefficients $a_1, a_2, \ldots, a_{k+1}$ are known:

$$u^2 = \sum_{k=1}^{\infty} b_k\eta^{k-1}, \quad u^3 = \sum_{k=1}^{\infty} c_k\eta^{k-1}. \tag{9}$$

If Eq. (6) is to be satisfied for all values of η, all the bracketed coefficients in Eq. (8) must vanish. This condition gives the recurrence relation:

$$a_{k+4} = (C_2a_{k+2} - C_3a_k - (\alpha a_k + \beta b_k + \gamma c_k - \omega^2 a_k)/(\eta\pi)^4)/C_1, \quad k = 1, 2, \ldots \tag{10}$$

which permits the computation of a_5 and higher coefficients based on the values of the four fundamental constants a_1, a_2, a_3, a_4 for a specified value of the frequency ω.

Each mode of vibration is considered separately by assigning a value for the mode number n, (for the first mode, $n = 1$, etc.). Consequently, the value of a_4 is set to zero because the cubic term involves the participation of higher modes in the basis of the solution. The remaining coefficients will, therefore, depend on the values of a_2 and the vibration frequency ω which appears as an auxiliary parameter. The true value of ω can be determined by invoking the Rayleigh energy principle, which states that for a conservative system, the maximum kinetic and strain energies, T_{max}, U_{max}, respectively, are equal. For the system considered,

$$T_{max} = \frac{1}{2}\omega^2 \int_0^1 u^2 \, \mathrm{d}x, \tag{11}$$

$$U_{max} = \int_0^1 \left[\frac{1}{2}\left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\right)^2 + \frac{1}{2}\alpha u^2 + \frac{1}{3}\beta u^3 + \frac{1}{4}\gamma u^4 \right] \mathrm{d}x. \tag{12}$$

In the above integrals, each term is written as a power series of η and the curvature can be shown to be

$$\begin{aligned} \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} &= (\eta\pi)^2 \left[(1-\eta^2)\frac{\mathrm{d}^2 u}{\mathrm{d}\eta^2} - \eta \frac{\mathrm{d}u}{\mathrm{d}\eta} \right] \\ &= (\eta\pi)^2 \sum_{k=1}^{\infty} [k(k+1)a_{k+2} - (k-1)^2 a_k]\eta^{k-1}. \end{aligned} \tag{13}$$

The evaluation of the integrals requires integration with respect to η. Since $|\eta| \leqslant 1$, the binomial theorem can be used to express dx, with the aid of Eq. (5), as a power series of η as follows:

$$\mathrm{d}x = \frac{\mathrm{d}\eta}{\eta\pi\sqrt{1-\eta^2}} = (r_1 + r_2\eta + r_3\eta^2 + r_4\eta^3 + \cdots)\,\mathrm{d}\eta, \tag{14}$$

where the constants r_i are given by

$$r_1 = \frac{1}{\eta\pi}, \quad r_{2k+1} = \frac{1}{\eta\pi k!}\prod_{i=1}^{k}\left(i - \frac{1}{2}\right), k \geq 1$$

and all the odd power coefficients are zero. The same limits of integration can conveniently be used which then covers a part of the beam.

3. Results and discussion

The first three non-linear normal modes and frequencies were computed by using the recurrence relation (10) in conjunction with the Rayleigh energy principle. For each mode, the amplitude–frequency dependence was investigated by assigning the appropriate value for n. For a specified amplitude of vibration as set by assigning a value for a_2, a frequency search was made for the natural frequency by computing a motion associated with a specified frequency ω from Eq. (10). This motion was then used to evaluate the kinetic and strain energies from Eqs. (11) and (12), respectively. The error function $\varepsilon = U_{max} - T_{max}$, which depends on ω always crossed the frequency axis at two frequencies for which the Rayleigh energy principle was satisfied. This feature of multiple solutions characterizes non-linear systems and the lower frequency was selected as the natural frequency of vibration The associated motion coefficients determine the corresponding normal mode.

To demonstrate the applicability of the present method to the linear vibration of beams, the formulation was first applied to the vibration of the beam without a foundation ($\alpha = \beta = \gamma = 0$).

Table 1
Frequency ratio ω/ω_L versus amplitude for the first three modes of the simply supported beam ($\alpha = 0$, $\gamma = 2000$)

Amplitude	Mode		
	First	Second	Third
0.10	1.0385	1.0018	1.0008
0.25	1.2057	1.0139	1.0029
0.50	1.6718	1.0525	1.0107
0.75	2.1480	1.1145	1.0233
1.00	2.7559	1.1969	1.0419

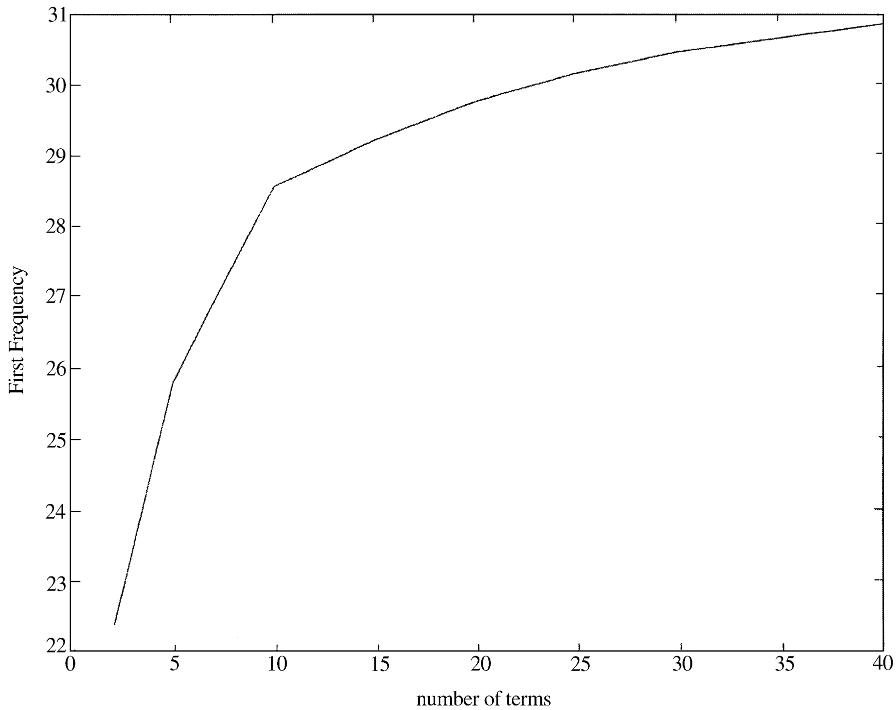

Fig. 1. Convergence of the fundamental frequency for amplitude = 1.15, $\alpha = 0, \gamma = 2000$.

The error function computed at the exact value of the first natural frequency (π^2) was less than 10E-14 when the first 40 terms were included in the power series. This number of terms was used in all subsequent computations.

In Table 1 the non-linear frequency ratio $\omega/\omega L$ of the first three modes is presented for various amplitudes and the specified values of the foundation parameters $\alpha = 0$ and $\gamma = 2000$. It can be seen that the fundamental frequency is significantly affected by the presence of the cubic non-linearity of the foundation. This influence is reduced at the higher modes with the third frequency being least affected. These results are supported by the findings of Qaisi [4]. In Fig. 1 the convergence of the fundamental frequency is demonstrated for amplitude of 1.15, $\alpha = 0, \gamma = 2000$.

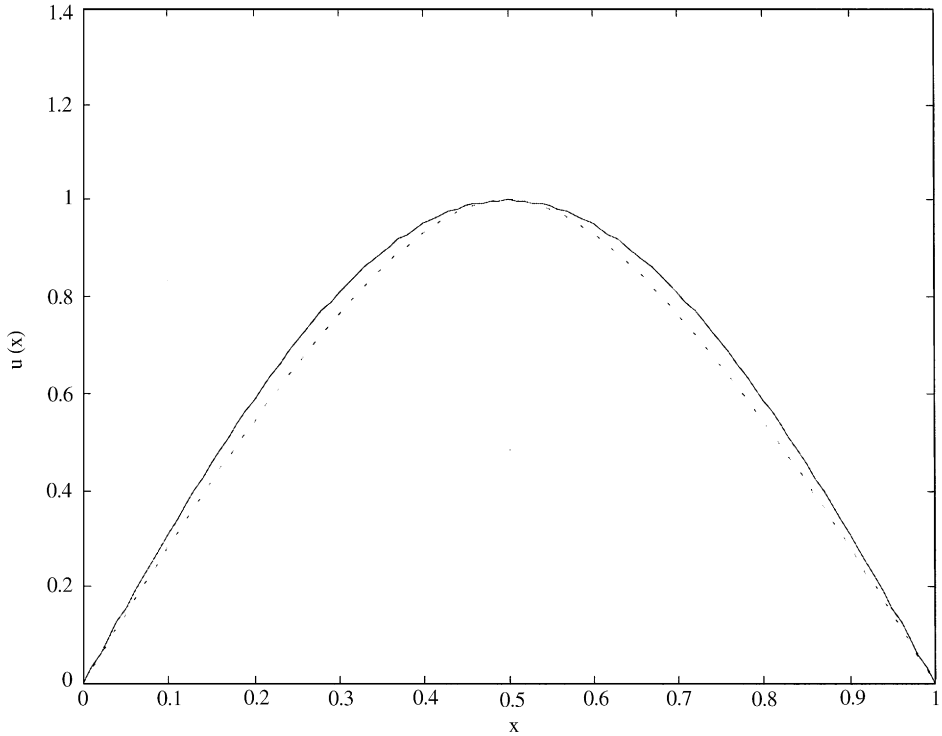

Fig. 2. The effect of the cubic non-linearity on the fundamental mode shape at unit amplitude (—, linear; ......, $\gamma = 2000$).

Table 2
First 30 power series coefficients for amplitude unity $\alpha = 0, \gamma = 2000$

k	1	2	3	4	5	6
a_k	0	9.10000e−1	0	0	0	4.7699e−2
a_{k+6}	0	2.0194e−2	0	1.1012e−2	0	6.6032e−3
a_{k+12}	0	4.6411e−3	0	2.9463e−3	0	2.1182e−3
a_{k+18}	0	1.5770e−3	0	1.2076e−3	0	9.6427e−4
a_{k+24}	0	7.5589e−4	0	6.1371e−4	0	5.0524e−4

In Fig. 2 the effect of the cubic non-linearity on the first normal mode is depicted for a vibration amplitude of unity. The cubic non-linearity is seen to increase the curvature of the beam at large amplitudes. The second and third modes are almost unaffected by the presence of the foundation even at large amplitudes, since they are indistinguishable from the linear modes. In Table 2 the first 30 power series coefficients are shown for a unit amplitude with $\alpha = 0, \gamma = 2000$. A progressive decrease in the value of the coefficients is seen which characterizes a convergent solution.

4. Conclusion

An analytical solution based on the power series method is presented for the free vibration of simply supported beams resting on non-linear foundations. A convergent solution was obtained by transforming the independent space variable into a new variable that satisfied the boundary conditions. The results show that, for a first order approximation, the quadratic non-linearity has no effect on the beam-free vibrations, whereas the cubic non-linearity strongly affects the beam fundamental frequency at large amplitudes with noticeable effect on the first mode shape. Higher modes and frequencies, however, are much less affected by the foundation.

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